Modelling Atwood's Machine with Three Degrees of Freedom

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An Atwood machine is a well-known device that consists of two bodies of different masses m_1, m_2 attached to opposite ends of a massless inextensible thread wound round a massless frictionless pulley (see Ref. [1]). It is assumed that each body can move only along a vertical, and the thread doesn't slip on the pulley. Such Atwood's machine is a simple mechanical system with one degree freedom that is usually used in the course of physics for demonstration of the uniformly accelerated motion of the system.

However, it is very difficult in practice to attain such a simple translational motion and the oscillations of the bodies inevitably arise. These oscillations may modify the system motion significantly and so the swinging Atwood machine has been a subject of a number of papers (see, for example, Refs. [2, 3, 4, 5, 6]). In particular, it has been proven that the system of differential equations describing dynamics of swinging Atwood's machine is not integrable, in general. It has been shown also that, depending on the mass ratio m_2/m_1, the system can demonstrate different types of motion, namely, periodic, quasi-periodic, or chaotic motion.

To clarify the physical reasons of such influence of oscillation on the system motion in the previous paper [7] we considered the simplest generalization of the Atwood machine when only one body of mass m_1 is allowed to swing in a plane while the other body of mass $m_2 > m_1$ can move only along a vertical. We have shown that oscillation results in increasing of the averaged thread tension which depends on the amplitude of oscillation. If increase of the averaged tension exceeds $(m_2 - m_1)g$, where g is a gravity acceleration, the body of smaller mass m_1 can pull the body m_2 up what is not possible in the system without oscillation.

In the present paper we consider the more complicated Atwood machine when both bodies are allowed to swing in the plane. Such a system has three degrees of freedom and can demonstrate different kinds of quasi-periodic motion depending on the masses difference and initial conditions. However, the equations of motion become more complicated and their analysis requires to combine symbolic and numerical calculations. We demonstrate here that such analysis can be successfully done with the computer algebra system Mathematica (see Ref. [8]) that is used for doing all relevant calculations and visualization of results.

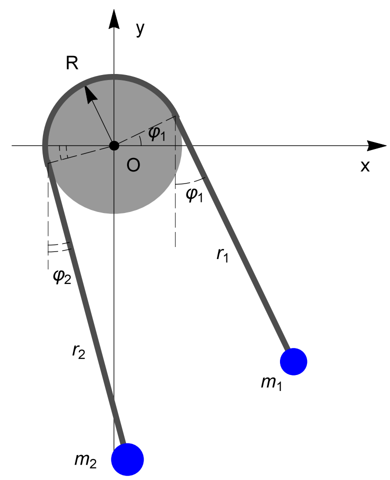

Figure 1: Atwood's machine with three degrees of freedom.

1 Equations of Motion

We consider a generalized model of the simple Atwood machine when both bodies are allowed to swing in the plane (see Fig. 1). Such a system has three degrees of freedom and its geometrical configuration can be described in terms of three variables, for example, two angles φ_1 and φ_2 determining deviations of the thread from the vertical and a length r of the thread between the body m_1 and the point, where the thread departs from the pulley in case of $\varphi_1 = 0$. Note that a length of the thread between the body m_2 and the point, where the thread departs from the pulley, is given by $(L - \pi R - r - R\varphi_2)$, where L is the length of the thread and R is a radius of the pulley.

The Lagrangian of the system can be written in the form

$$\begin{aligned}\mathscr{L} = \frac{(m_1+m_2)R^2+I_0}{2R^2}\dot{r}^2 + \frac{m_1}{2}\left(r+R\varphi_1\right)^2\dot{\varphi}_1^2 \\ +\frac{m_2}{2}\left(L-r-\pi R-R\varphi_2\right)^2\dot{\varphi}_2^2 - m_1 g\left(R\sin\varphi_1-(r+R\varphi_1)\cos\varphi_1\right) \\ +m_2 g\left(R\sin\varphi_2+(L-r-\pi R-R\varphi_2)\cos\varphi_2\right)\ ,\end{aligned} \tag{1}$$

where the dot denotes differentiation with respect to time, and I_0 is a moment of inertia of the pulley. Using Eq. (1) and doing standard symbolic calculations, we obtain the equations of motion in the form

$$\kappa\ddot{r} = g(\cos\varphi_1 - \mu\cos\varphi_2) + (r+R\varphi_1)\dot{\varphi}_1^2 - \mu(L-r-\pi R-R\varphi_2)\dot{\varphi}_2^2\ , \tag{2}$$

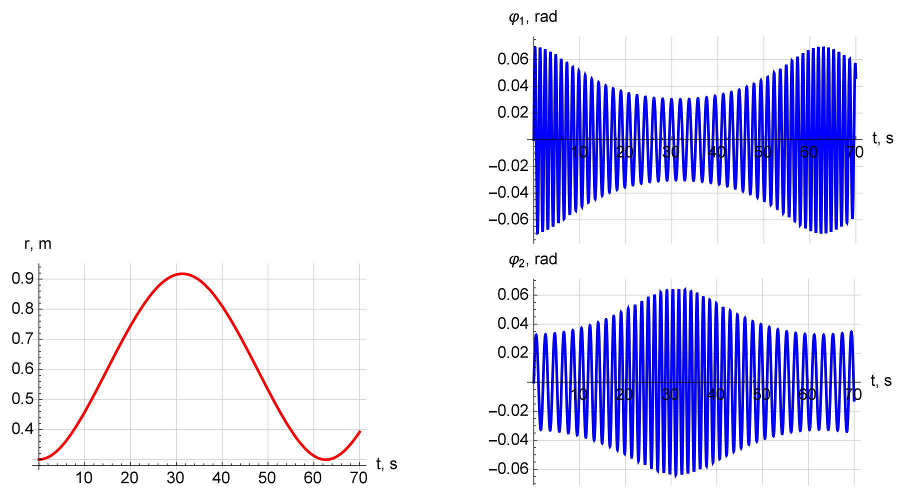

Figure 2: Motion of the Atwood machine in case of $m_1 = m_2$.

$$(r + R\varphi_1)\ddot{\varphi}_1 = -g\sin\varphi_1 - 2\dot{r}\dot{\varphi}_1 - R\dot{\varphi}_1^2\,, \tag{3}$$

$$(L - r - \pi R - R\varphi_2)\ddot{\varphi}_2 = -g\sin\varphi_2 + 2\dot{r}\dot{\varphi}_2 + R\dot{\varphi}_2^2\,, \tag{4}$$

where $\mu = m_2/m_1$,

$$\kappa = \frac{I_0 + (m_1 + m_2)R^2}{m_1 R^2}\,.$$

2 Main result

One can readily check that equations of motion (2)-(4) cannot be solved symbolically. However, choosing some realistic values of the system parameters, we can obtain the corresponding numerical solution for different initial conditions and analyze motion of the system.

As an example, let us consider the case of equal masses ($m_1 = m_2$) and assume that the bodies are at rest. If the body of mass m_1 gets a small horizontal initial velocity it starts to oscillate. As a result an average value of the thread tension becomes greater than the gravity force $m_2 g$ and the oscillating body starts to move down and pull up the second body (see [7]). However, if both bodies being at rest get different horizontal initial velocities then both of them start to oscillate with different amplitudes. Solving Eqs. (2)-(4) with the initial conditions $\varphi_1(0) = \varphi_2(0) = \dot{r}(0) = 0$, $r(0) = 0.3$, $\dot{\varphi}_1(0) = 0.4$, $\dot{\varphi}_2(0) = 0.1$, for instance, we obtain a solution shown in Fig. 2.

One can readily see that initially the body of mass m_1 oscillates with the amplitude being greater than that of the body m_2. Consequently, the thread tension in the right-hand side of the system is greater than in the left-hand side and the body m_1 moves down and pull up the body m_2. However, a length of the thread between the body m_1 and the pulley increases and amplitude of its oscillation decreases while amplitude of the body m_2 oscillation grows up. Finally, an average tension of the thread between the body m_2 and the pulley becomes greater that the tension in the right-hand side of the system. As a results the pulley stops and then starts to rotate in opposite direction. Then the roles of the bodies change and the system continues its motion. Thus, due to oscillations of the bodies the system demonstrates quasi-periodic motion which is not possible in case of the classical Atwood machine with bodies of equal masses.

3 Conclusions

In the present talk we have demonstrated an influence of oscillation on the Atwood machine motion in the case when both bodies are allowed to oscillate in a plane. Simulating motion of such Atwood's machine with the computer algebra system Wolfram Mathematica, we have shown that even small oscillations can completely modify its motion, while the simple Atwood machine demonstrates only the uniformly accelerated motion of the bodies. Note that such simulation promotes development of physical intuition and better understanding of the subject.

References

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